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From Effect Algebras to Algebras of Effects (Sum Brouwer–Zadeh Algebras)[†]

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The algebraic structures arising in the axiomatic framework of unsharp quantum mechanics based on effect operators on a Hilbert space are investigated. It is stressed that usually considered "effect" algebras neglect the unitary Brouwerian map of complementation, and the main results based on this complementation are collected, showing the enrichment produced into the theory by its introduction. In particular, in these structures two notions of sharpness can be considered: *K*-sharpness induced by the usual complementation of effect algebras and *B*-sharpness induced by this new complementation. Quantum (resp., classical) SBZ algebras are then characterized by the condition of *B*-coherence (resp., *B*-coherence plus *B*-compatibility), showing that in this case the poset of all *B*-sharp elements is orthomodular (resp., Boolean algebra). In the unsharp context of effect operators, the finite dimensionality of the Hilbert space or the finiteness of a von Neumann algebra are both characterized by a de Morgan property of the Brouwer complementation. Moreover, since effect operators on a pre-Hilbert space give rise to a standard model of effect algebras, a characterization of completeness of pre-Hilbert spaces is given making use of the Brouwer complement.

1. SUM BROUWER–ZADEH (BZ) EFFECT ALGEBRAS

Sum Brouwer–Zadeh (SBZ) algebras have been introduced [4] in order to give an axiomatic algebraic framework to standard unsharp quantum mechanics (UQM) based on effect operators on complex Hilbert spaces. It essentially consists of a *regular effect algebra*:

$$\langle \mathcal{E}, \oplus, ', 0, 1 \rangle$$

which is an effect algebra as defined in refs. 15 and 17 (equivalently formu-

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lated some years before in ref. 20 as *unsharp orthoalgebra*, and called *weak orthoalgebra* in ref. 15) such that, denoting by $a \perp b$ the fact that $a \oplus b$ is defined, the following *regularity* condition holds:

$$a \perp a \quad \text{and} \quad b \perp b \qquad \text{imply} \quad a \perp b$$

This regular effect algebra is furthermore equipped with another unary operation $^{\sim}\colon \mathscr{E} \mapsto \mathscr{E}$, called the *Brouwer complementation*, satisfying the following conditions:

(**boc-1**) [*B-Symmetry law*]. Let $a, b \in \mathscr{E}$. Then,

$$\exists r\colon \quad a \oplus r = b^{\sim} \qquad \text{implies} \qquad \exists s\colon \qquad b \oplus s = a^{\sim}$$

(**boc-2**) [*B-Orthogonality law*]:

$$\forall a \in \mathscr{E}, \quad a \perp a^{\sim} \qquad [\text{i.e.,} \quad \exists a \oplus a^{\sim}]$$

(**boc-3**) [*B-Noncontradiction law*]. Let $a, c \in \mathscr{E}$. Then,

$$\exists r\colon \quad a^{\sim} \oplus r = c \quad \text{and} \quad \exists s\colon \quad a^{\sim\sim} \oplus s = c \qquad \text{imply} \quad c = 1$$

Strangely enough, the possibility of working with a more articulate structure, with the corresponding enrichment of the theory, has been neglected by the scientific community involved in axiomatic UQM. In this paper the main results based on this complementation are collected.

In particular, two notions of sharpness can be introduced in these structures: *K-sharpness* induced by the Kleene complementation $'\colon \mathscr{E} \mapsto \mathscr{E}$ usually obtained from any effect algebras, and *B-sharpness* induced by the new Brouwer complementation. In Section 4 *quantum* SBZ algebras can be defined by the condition of *B*-coherence, showing that in this case, differently from the notion of *K*-coherence introduced by Foulis and Bennett [17] (and assuring that the *whole* effect algebra is orthomodular), only the substructure of all *B*-sharp elements is an orthomodular poset, whereas in general the larger effect algebra is not so. Of course, this is the case of the concrete algebra of effect operators on a Hilbert space. A similar result is given for the case of *classical* SBZ algebras, i.e., SBZ algebars satisfying both the *B*-coherence and the *B*-completeness laws, whose substructure of *B*-sharp elements is a Boolean algebra.

As in the sharp case, the modularity of the lattice of projectors characterizes the finite dimensionality of the Hilbert space; in Section 5 it is recalled that in the unsharp context of effect operators the de Morgan property of the Brouwer complement characterizes the finite dimensionality of the Hilbert space (and in the case of von Neumann algebras, its finiteness).

We want to stress that regular effect algebras (without the $^{\sim}$ operation) admit as standard models the collection of all effect operators on a *pre-*

Hilbert space. In Section 6, making use of a result of Gudder [21], we give a characterization of completeness of a pre-Hilbert space in which this new unary operation is strongly involved.

1.1. Standard Models of SBZ Algebras

In order to show that the definition of SBZ algebra is not empty, let us consider some interesting examples.

Example 1.1. Usual *fuzzy set theory on the universe U*. Let U be a nonempty set. Let $\mathscr{E}(U) := [0, 1]^U$ be the collection of all mappings $f\colon U \mapsto [0, 1]$ (*fuzzy sets* or *generalized characteristic functionals* on the universe U); in particular, $\underline{1}$ (resp., $\underline{0}$) is the fuzzy set associating with any $x \in U$ the number $\underline{1},(x) := 1$ (resp, 0). Then the structure $\langle [0, 1]^U, \perp, \oplus, ', {}^\sim, \underline{0}, \underline{1} \rangle$ is an SBZ-algebra with respect to:

(1) The orthogonality relation: let $f, g \in [0, 1]^U$; then $f \perp g$ iff $\forall x \in U$, $(f + g)(x) \leq 1$.
(2) The partial sum operation: let $f, g \in [0, 1]^U$ be such that $f \perp g$; then $f \oplus g := f + g$.
(3) The K-complementation: $\forall f \in [0, 1]^U$, then $f' = \underline{1} = f$ (in particular, $\underline{0} = \underline{1}'$).
(4) The B-complementation: $\forall f \in [0, 1]^U$; then $f^\sim = \chi_{A_0(f)}$ [where $A_0(f) := \{x \in U\colon f(x) = 0\}$ is the *certainly-no domain* of f, and for every subset A of U, $\chi_A(x)$ is the characteristic functional of A; in particular, $\underline{0} = \chi_\phi$ and $\underline{1} = \chi_U$].

Example 1.2. Standard *unsharp QM on the Hilbert space* $\mathscr{H}$. Let $\mathscr{H}$ be a complex, separable Hilbert space. Let $\mathscr{E}(\mathscr{H})$ be the collection of all linear operators $F\colon \mathscr{H} \mapsto \mathscr{H}$ such that the corresponding *probability distribution function* $\forall \psi \in \mathscr{H}\backslash\{\mathbf{0}\}$, $f_F(\psi) := (\psi|F\psi)/\|\psi\|^2$, is a fuzzy set on the universe $\mathscr{H}' := \mathscr{H}\backslash\{\mathbf{0}\}$ (equivalently, iff $\forall \psi \in \mathscr{H}$, $0 \leq \langle \psi|F\psi\rangle \leq \|\psi\|^2$). These operators are called *effect operators*; in particular, $\mathbb{I}$ (resp., $\mathbb{O}$) is the effect operator associating with any $\psi \in \mathscr{H}$ the vector $\mathbb{I}(\psi) := \psi$ (resp., $\mathbf{0}$). Then the structure $\langle \mathscr{E}(\mathscr{H}), \perp, \oplus, ', {}^\sim, \mathbb{O}, \mathbb{I} \rangle$ is an SBZ-algebra with respect to:

(1) The orthogonality relation: let $F, G \in \mathscr{E}(\mathscr{H})$; then $F \perp G$ iff $\forall \psi \in \mathscr{H}$, $\langle \psi|(F + G)\psi\rangle \leq \|\psi\|^2$.
(2) The partial sum operation: let $F, G \in \mathscr{E}(\mathscr{H})$ be such that $F \perp G$; then $F \oplus G := F + G$.
(3) The K-complementation: $\forall F \in \mathscr{E}(\mathscr{H})$, $F := \mathbb{I} - \mathrm{F}$ (in particular $\mathbb{O} = \mathbb{I}'$).
(4) The B-complementation: $\forall F \in \mathscr{E}(\mathscr{H})$, $F^\sim := \mathbb{P}_{M_0(F)}$ [where $M_0(F) := \{\psi \in \mathscr{H}\colon \langle \psi|F\psi\rangle = 0\} = Ker\,(F)$ is the *certainly-no subspace*

of F, and for every subspace M of $\mathcal{H}$, $\mathbb{P}_M$ is the orthogonal projection on M; in particular, $\mathbb{O} = \mathbb{P}_{\{\mathbf{0}\}}$ and $\mathbb{I} = \mathbb{P}_{\mathcal{H}}$].

Let us notice that in the case of a pre-Hilbert space $\mathcal{H}$, one can also consider the collection of all effect operators defined formally as in the Hilbert space case. The unique difference is that only points (1)–(3) can be considered, obtaining a structure $\langle \mathcal{H}, \perp, \oplus, {}'\mathbb{O},, \mathbb{I}\rangle$ which is only of effect algebra.

Example 1.3. Unsharp QM on von Neumann algebras. Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space with self-adjoint part $\mathcal{M}^s$ and positive cone $\mathcal{M}^+$. Then on the unit interval $\mathcal{E}(\mathcal{M}) = [0, 1]$ of this von Neumann algebra, points (1)–(4) of Example 1.2 are well defined and the resulting structure is of SBZ poset [10].

Let us notice that if we consider the larger category of C^*-algebras, then also in this case only points (1)–(3) are well defined, obtaining that the structure of the unit interval is only of an effect algebra.

2. DISTINCTION BETWEEN SHARP AND UNSHARP ELEMENTS

From any SBZ algebra $\langle \mathcal{E}, \perp, \oplus, ', {}^\sim, 0, 1\rangle$ it is possible to induce a BZ poset structure bounded by the *least* element 0 and the *greatest* element 1 [14],

$$\langle \mathcal{E}, \leq, ', {}^\sim, 0, 1\rangle$$

where the binary relation $\leq$ on $\mathcal{E}$ is the usual effect algebra partial ordering [17]:

$$a \leq b \qquad \text{iff} \quad \exists c\colon \quad a \oplus c = b \tag{2.1}$$

The mapping $'\colon \mathcal{E} \mapsto \mathcal{E}$ is the *Kleene complementation* satisfying:

(K1) $a = a''$.
(K2) $a \leq b$ implies $b' \leq a'$.
(K3) $a \leq a'$ and $b' \leq b$ imply $a \leq b$.

[In general the "*noncontradiction*" law ($\forall a \in \mathcal{E}$, $a \wedge a' = 0$) and the "*excluded-middle*" law ($\forall a \in \mathcal{E}$, $a \vee a' = 1$) do not hold; note that under conditions (K1) and (K2) these two laws are mutually equivalent.]

The mapping $^\sim\colon \mathcal{E} \mapsto \mathcal{E}$ is the *Brouwer complementation* satisfying:

(B1) $a \leq a^{\sim\sim}$.
(B2) $a \leq b$ implies $b^\sim \leq a^\sim$.
(B3) $a \wedge a^\sim = 0$.

[In general the "*strong double negation*" law ($\forall a \in \mathscr{E}$, $a = a^{\sim\sim}$) does not hold.]

In these BZ structures one can single out two kinds of sharp elements: the *K-sharp*, for which the excluded-middle law for the Kleene complementation holds, $\mathscr{E}_K = \{h \in \mathscr{E}\colon h \vee h' = 1\}$, and the *B-sharp*, for which the strong double negation law for the Brouwer complementation holds, $\mathscr{E}_B = \{\alpha \in \mathscr{E}\colon \alpha = \alpha^{\sim\sim}\}$. Trivially, $\mathscr{E}_B \subseteq \mathscr{E}_K$.

Proposition 2.1. The set $\mathscr{E}_K$ of all *K*-sharp elements:

1. Is closed with respect to the Kleene complementation: $\forall h \in \mathscr{E}_K$, $h' \in \mathscr{E}_K$.
2. Equipped with the restriction of the partial order relation (2.1), turns out to be an orthoposet (i.e., a poset with standard orthocomplementation),

$$\langle \mathscr{E}_K, \leq, {}', 0, 1\rangle$$

Proposition 2.2. The set $\mathscr{E}_B$ of all *B*-sharp elements:

1. Is closed with respect to the $\oplus$ operation: let $\alpha, \beta \in \mathscr{E}_B$ be such that $\exists \alpha \oplus \beta$ in $\mathscr{E}$, then $\alpha \oplus \beta \in \mathscr{E}_B$.
2. The two complementations collapse on its elements ($\forall \alpha \in \mathscr{E}_B$, $\alpha' = \alpha^{\sim}$), and is closed with respect to this (unique and standard) orthocomplementation ($\forall \alpha \in \mathscr{E}_B$, $\alpha' \in \mathscr{E}_B$).
3. Is an orthoalgebra $\langle \mathscr{E}_B, \oplus, {}', 0, 1\rangle$ [18], whose induced partial order structure $\langle \mathscr{E}_B, \leq, {}', 0, 1\rangle$ is an orthoposet, contained in the orthoposet $\langle \mathscr{E}_K, \leq, {}', 0, 1\rangle$ of all *K*-sharp elements.

Example 2.1. Let us consider the SBZ algebra of fuzzy sets on U discussed in Example 1.1. Then, the partial order (2.1) induced from the partial sum operation (2) is the pointwise ordering

$$\forall f, g \in \mathscr{E}(U), \quad f \leq g \qquad \text{iff} \qquad \forall x \in U, \quad f(x) \leq g(x) \tag{2.2}$$

With respect to this partial ordering $\mathscr{E}(U)$ can be characterized as the set of all real-valued mappings f defined on U such that $\underline{0} \leq f \leq \underline{1}$.

We have the following result:

Theorem 2.1. Let U be a universe space and let $[0, 1]^U$ be the SBZ algebra of all fuzzy sets on U. Then:

1. The induced structure $\langle [0, 1]^U, \leq, {}', {}^{\sim}, \underline{0}, \underline{1}\rangle$ is a distributive BZ complete lattice.
2. The sets of *B*-sharp and *K*-sharp elements coincide, and in turn they coincide with the Boolean (atomic, complete) lattice $\{0, 1\}^U$ of all characteristic functionals:

$$([0, 1]^U)_B = ([0, 1]^U)_K = \{0, 1\}^U$$

Let us stress that the Boolean lattice $\{0, 1\}^U$ of all (0,1-valued) characteristic functionals (B- and K-sharp fuzzy sets) on U is isomorphic to the Boolean lattice $\mathcal{P}(U)$ of all subsets of U (the *power set* of U) by the one-to-one correspondence A_1: $\{0, 1\}^U \mapsto \mathcal{P}(U)$ associating with any characteristic functional $\chi: U \mapsto \{0, 1\}$ its *certainly-yes domain* $A_1(\chi) := \{x \in U: \chi(x) = 1\}$.

Therefore, B- and K-sharp fuzzy sets (characteristic functionals) are identifiable with subsets of the universe U.

Example 2.2. In the case of the SBZ algebra $\mathcal{E}(\mathcal{H})$ of effect operators on the Hilbert space $\mathcal{H}$, the partial order (2.1) assumes the form

$$\forall F, G \in \mathcal{E}(\mathcal{H}), \quad F \leq G \qquad \text{iff} \qquad \forall \psi \in \mathcal{H}, \quad \langle \psi | F\psi \rangle \leq \langle \psi | G\psi \rangle \tag{2.3}$$

Note that two effect operators are in the relation $F \leq G$ iff the corresponding fuzzy sets are in the relation $f_F \leq f_G$. Also, in this example $\mathcal{E}(\mathcal{H})$ can be characterized as the set of all linear operators F on $\mathcal{H}$ such that $\mathbb{O} \leq F \leq \mathbb{I}$.

In the present Hilbert space case we have the following result.

Theorem 2.2. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C}\mathrm{E}(\mathcal{H})$ be the SBZ algebra of all effect operators on $\mathcal{H}$. Then:

1. The induced structure $\langle \mathcal{E}(\mathcal{H}), \leq, ', {}^\sim, \mathbb{O}, \mathbb{I} \rangle$ is a BZ poset which is not a lattice.
2. The sets of B-sharp and K-sharp elements coincide, and in turn they coincide with the orthomodular (atomic, complete) lattice $\Pi(\mathcal{H})$ of all orthogonal projections

$$\mathcal{E}_B(\mathcal{H}) = \mathcal{E}_K(\mathcal{H}) = \Pi(\mathcal{H})$$

Also, in this Hilbert space case, the orthomodular lattice $\Pi(\mathcal{H})$ of all orthogonal projections (B- and K-sharp effects) on $\mathcal{H}$ is isomorphic to the orthomodular lattice $\mathcal{M}(\mathcal{H})$ of all subspaces of $\mathcal{H}$, by the one-to-one correspondence M_1: $\Pi(\mathcal{H}) \mapsto \mathcal{M}(\mathcal{H})$ associating with any orthogonal projection P: $\mathcal{H} \mapsto \mathcal{H}$ its *certainly-yes subspace* $M_1(P) := \{\psi \in \mathcal{H}: \langle \psi | P\psi \rangle = \|\psi\|^2\} = Ker(\mathbb{I} - P)$.

Therefore, B- and K-sharp effect operators (orthogonal projections) are identifiable with subspaces of the Hilbert space $\mathcal{H}$.

2.1. Sharpness in the BZ Structure of Special Elements

In the context of the algebraic BZ poset structure $\langle \mathcal{E}, \leq, ', {}^\sim, 0, 1 \rangle$, the 0-*kernel* and 1-*kernel* of the Kleene complementation are defined, respectively, as the two subsets

$$N_0'(\mathscr{E}) := \{a \in \mathscr{E}: a \leq a'\} \qquad \text{and} \qquad N_1'(\mathscr{E}) := \{b \in \mathscr{E}: b' \leq b\}$$

Setting $\hat{N}_0'(\mathscr{E}) := N_0'(\mathscr{E}) \setminus \{0\}$ and $\hat{N}_1'(\mathscr{E}) := N_1'(\mathscr{E}) \setminus \{1\}$, we call the elements of $\mathscr{E}^{(se)} := \mathscr{E} \setminus (\hat{N}_0'(\mathscr{E}) \cup \hat{N}_1'(\mathscr{E}))$ *special*.

Theorem 2.3. Let $\mathscr{E}$ be BZ poset. Then the following hold:

1. The set of all special effects is not empty since it contains all B-sharp elements: $\mathscr{E}_B \subseteq \mathscr{E}^{(se)}$.
2. $\mathscr{E}^{(se)}$ is closed with respect to both Kleene and Brouwer complementations ($\forall a \in \mathscr{E}^{(se)}$: a', $a^\sim \in \mathscr{E}^{(se)}$), and so equipped with the restriction $\leq_{se}$ of the partial order (2.1) and the complementations $'$: $\mathscr{E}^{(se)} \mapsto \mathscr{E}^{(se)}$ and $^\sim$: $\mathscr{E}^{(se)} \mapsto \mathscr{E}^{(se)}$ determines a BZ poset in which both the excluded-middle principle ($\forall f \in \mathscr{E}^{(se)}$, $f \vee_{se} f' = 1$) and the noncontradiction principle ($\forall f \in \mathscr{E}^{(se)}$, $f \wedge_{se} f' = 0$) for the Kleene complementation hold, i.e., $'$ is a standard orthocomplementation.

Proof. Conditions $a = a^{\sim\sim} (a \in \mathscr{E}_B)$ and $a \leq a' [a \in N_0'(\mathscr{E})]$ imply $a = a^{\sim\sim} \leq a'$, from which, applying the Kleene complementation, we obtain $a = a'' \leq a^{\sim\sim\prime}$, that is, $(a^{\sim\sim}) \leq (a^{\sim\sim})'$. Let us recall that from $a = a^{\sim\sim}$ it follows that $(a^{\sim\sim}) = (a^{\sim\sim})^{\sim\sim}$, i.e., $a^{\sim\sim} \in \mathscr{E}_B$, and so, using points 2 and 3 of Proposition 2.2, we have $(a^{\sim\sim}) = (a^{\sim\sim}) \wedge (a^{\sim\sim})' = 0$, concluding that $a = a^{\sim\sim} = 0$.

Analogously, conditions $b \in \mathscr{E}_B$ and $b \in N_1'(\mathscr{E})$ imply $b = 1$.

Let now $f \in \mathscr{E}^{(se)}$ be a special effect and let $g \in \mathscr{E}^{(se)}$ be a lower bound of $\{f, f'\}$ with respect to $\leq_{se}$, i.e., $g \leq_{se} f$ and $g \leq_{se} f'$. Then, $g \leq_{se} f \leq_{se} g'$, i.e., $g = 0$. Note that in general the latter conclusion cannot be inferred if we consider the special effect $f \in \mathscr{E}^{(se)}$ inside the wide Kleene poset $\mathscr{E}$; indeed, if we take an element $g \in \mathscr{E}$ such that $g \leq f$ and $g \leq f'$, then we can only infer that $g \leq g'$, i.e., $g \in N_0'(\mathscr{E})$. ■

If, as usual, one defines as K-sharp elements of $\mathscr{E}^{(se)}$ the collection $\mathscr{E}_K^{(se)}$ of all special effects for which the excluded middle and the noncontradiction laws hold for the complementation $'$ and as B-sharp elements the collection $\mathscr{E}_B^{(se)}$ of all special effects for which the strong double negation law holds for the complementation$^\sim$; then one obtains

$$\mathscr{E}_B^{(se)} \subseteq \mathscr{E}_K^{(se)} = \mathscr{E}^{(se)}$$

That is, in the case of the BZ poset of special elements the notion of K-sharpness is meaningless.

Let us recall that the notion of special effects in quantum mechanics was introduced for the first time by Garola [19] (see also ref. 2) in order to avoid the presence of some physically unpleasant effect operators.

Example 2.3. In the standard BZ poset structure of effect operators on a Hilbert space $\mathcal{H}$, $\langle \mathcal{E}(\mathcal{H}), \leq, {}^{\sim}, {}', 0, 1\rangle$, an effect $F \in \mathcal{E}(\mathcal{H})$ is special iff there exist nonzero vectors $\psi, \varphi \in \mathcal{H}\backslash\{\mathbf{0}\}$ such that

$$\langle\psi|F\psi\rangle < \tfrac{1}{2}\|\psi\|^2 \qquad \text{and} \qquad \langle\varphi|F\varphi\rangle > \tfrac{1}{2}\|\varphi\|^2 \tag{2.4}$$

In this Hilbert space case of special effects one obtains the following inclusions:

$$\Pi(\mathcal{H}) = \mathcal{E}_B^{(se)}(\mathcal{H}) \subset \mathcal{E}_K^{(se)}(\mathcal{H}) = \mathcal{E}^{(se)}(\mathcal{H})$$

That is, orthogonal projections are the *B*-sharp elements, whereas according to the general theory, *K*-sharp elements coincide with the whole BZ structure.

The set $\mathcal{E}^{(se)}(\mathcal{H})$ has the particular "pathological" behavior that in general it is not closed with respect to the restriction of the partial sum operation defined in $\mathcal{E}(\mathcal{H})$. For instance, let us consider the two-dimensional Hilbert space $\mathbb{C}^2$. Then the two effect operators

$$\begin{pmatrix} 2/3 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 2/3 \end{pmatrix}$$

are special, but their sum $(2/3)\mathbb{I}$ is not special.

3. THE INDUCED ROUGH APPROXIMATION SPACE

Making use of the two complementations of the BZ poset structure induced from every SBZ algebra $\mathcal{E}$, it is possible to construct (according to refs. 3 and 5) an associated *rough approximation space* $\langle \mathcal{E}, \mathcal{E}_B, {}^{\nu}, {}^{\mu}\rangle$, where:

(r1) $\mathcal{E}$ is the set of *approximable* elements.

(r2) $\mathcal{E}_B$ is the set of *definable* (sharp or crisp) elements.

(r3) ${}^{\nu}: \mathcal{E} \mapsto \mathcal{E}_B$ is the *inner approximation* map associating with any element $a \in \mathcal{E}$ its inner sharp approximation $a^{\nu} := a'^{\sim}$ (the *necessity* of a).

(r4) ${}^{\mu}: \mathcal{E} \mapsto \mathcal{E}_B$ is the *outer approximation* map associating with any element $a \in \mathcal{E}$ its outer approximation $a^{\mu} := a^{\sim\prime}$ (the *possibility* of a).

The inner approximation a^{ν} is the best approximation of a from *the bottom* by *B*-sharp elements, since:

(in1) a^{ν} is *B*-sharp.

(in2) It is an approximation of a from the bottom: $a^{\nu} \leq a$.

(in3) It is the best *B*-sharp approximation of a from the bottom: $\beta \in \mathcal{E}_B$ and $\beta \leq a$ imply $\beta \leq a^{\nu}$.

The outer approximation a^μ is the best approximation of a from *the top* by B-sharp elements, since:

(ou1) a^μ is B-sharp.
(ou2) It is an approximation of a from the top: $a \leq a^\mu$.
(ou3) It is the best B-sharp approximation of a from the top: $\gamma \in \mathscr{E}_B$ and $a \leq \gamma$ imply $a^\mu \leq \gamma$.

From (in2) and (ou2) we have that, for every approximable element $a \in \mathscr{E}$, the two B-sharp elements $a^\nu, a^\mu \in \mathscr{E}_B$ are such that $a^\nu \leq a \leq a^\mu$. This gives rise to the *rough approximation* $r(a) = (a^\nu, a^\mu) \in \mathscr{E}_B \times \mathscr{E}_B$ of the element $a \in \mathscr{E}$, according to the Pavlak's approach to roughness [26]. Trivially, the rough approximation of a B-sharp element $\alpha \in \mathscr{E}_B$ is the identical pair $r(\alpha) = (\alpha, \alpha)$.

Let us stress that the above conditions (ou1–3) and (in 1–3) can be summarized in the following way:

(BSD_1) $\forall a \in \mathscr{E}, \exists a^\mu \in \mathscr{E}_B$: $a \leq a^\mu$ and if $\beta \in \mathscr{E}_B$ satisfies $a \leq \beta$, then $a^\mu \leq \beta$.
(BSD_2) $\forall a \in \mathscr{E}, \exists a^\nu \in \mathscr{E}_B$: $a^\nu \leq a$ and if $\gamma \in \mathscr{E}_B$ satisfies $\gamma \leq a$, then $\gamma \leq a^\nu$.

Conditions (BSD_1) and (BSD_2) are equivalent since the two approximation maps are in duality by the Kleene complementation: $\forall a \in \mathscr{E}$, $a^\nu = a'^{\mu\prime}$ and $a^\mu = a'^{\nu\prime}$. Borrowing some terminology from ref. 21, we can say that *SBZ algebras are always B-sharply dominating*.

Example 3.1. In the SBZ algebra $[0, 1]^U$ of all fuzzy sets on the universe U, for every fuzzy set f its necessity is the characteristic functional $f^\nu = \chi_{A_1(f)}$ of the *certainly-yes* domain $A_1(f) := \{x \in U: f(x) = 1\}$ of f, and its possibility is the characteristic functional $f^\mu = \chi_{A_0(f)^c}$ of the set-theoretic complement of the certainly-no domain of f. Note that $A_0(f)^c = A_p(f) := \{x \in U: f(x) \neq 0\}$, the latter being the *possibility* domain of f.

Therefore, the rough approximation of f can be identified with the ordered pair of subsets of U: $r(f) \equiv (A_1(f), A_0(f)^c)$.

Example 3.2. In the SBZ algebra $\mathscr{E}(\mathscr{H})$ of all effect operators on a Hilbert space $\mathscr{H}$, for every effect operator F the associated necessity is the projector $F^\nu = \mathbb{P}_{M_1(F)}$ onto the *certainly-yes* subspace $M_1(F) := \{\psi \in \mathscr{H}: \langle\psi|F\psi\rangle = \|\psi\|^2\} = \ker(\mathbb{I} - F)$ of F, and the corresponding possibility is the projector $F^\mu = \mathbb{P}_{M_0(F)^\perp}$ onto the orthocomplement of the certainly-no subspace of F.

Therefore, in the Hilbertian SBZ algebra the rough approximation of every effect operator F can be identified with the pair of subspaces of $\mathscr{H}$: $r(F) \equiv (M_1(F), M_0(F)^\perp)$.

4. QUANTUM AND CLASSICAL SBZ ALGEBRAS

In an SBZ effect algebra, besides the natural orthogonality relation "$a \perp b$ iff a $\oplus$ b exists" (which concides with the orthogonality relation induced from the Kleene complementation: $a \perp_K b$ iff $a \leq b'$, called *K-orthogonality*), one can introduce another orthogonality relation induced from the Brouwer complementation: $a \perp_B b$ iff $a \leq b^{\sim}$, called *B-orthogonality.* Trivially, $a \perp_B b$ implies $a \perp_K b$; moreover, $a \perp_B b$ iff $a^{\mu} \perp_K b^{\mu}$ (the B-orthogonality of a, b is just the K-orthogonality of their respective possibilities).

Definition 4.1. A *quantum* SBZ algebra is any SBZ algebra $\mathscr{E}$ satisfying the following *B-coherence law*:

(B) For any triple a, b, $c \in \mathscr{E}$ of pairwise B-orthogonal elements, written $\{a, b, c\} \perp_B$, there exists the sum $a \oplus b \oplus c \in \mathscr{E}$.

The above B-coherence law involved the B-orthogonality relation, in contrast to the Foulis–Bennett coherence law [17], which is a *K-coherence law* since it can be expressed as:

(K) For any triple a, b, $c \in \mathscr{E}$ of pairwise K-orthogonal elements, written $\{a, b, c\} \perp_K$, there exists the sum $a \oplus b \oplus c \in \mathscr{E}$.

Let us recall Theorem 5.3 of ref. 17:

An effect algebra is an orthomodular poset iff it satisfies the K-coherence law.

In contrast, if an SBZ effect algebra satisfies the B-coherence law, then we cannot state that it is an orthomodular poset (see the case of the SBZ algebra of effect operators on a Hilbert space discussed in Theorem 4.2 below, which satisfies the B-coherence law, but whose induced Kleene poset is not orthomodular).

The following result is given in ref. 4.

Theorem 4.1. Let $\mathscr{E}$ be an SBZ algebra satisfying the B-coherence law. Then, the set $\mathscr{E}_B$ of all B-sharp elements is an *orthomodular* orthoalgebra.

In the Hilbert space model one has the following further result.

Theorem 4.2. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{E}(\mathscr{H})$ be the SBZ algebra of all effect operators on $\mathscr{H}$. Then, the SBZ structure $\langle \mathscr{E}(\mathscr{H}), \oplus, ', {}^{\sim}, \mathbb{O}, \mathbb{I} \rangle$ is a quantum SBZ algebra.

Let us recall (see Theorem 2.2) that the induced structure $\langle \mathscr{E}(\mathscr{H}), \leq, ', {}^{\sim}, \mathbb{O}, \mathbb{I} \rangle$ is a BZ poset (which is not a lattice) whose sets of B-sharp and K-sharp elements coincide with the orthomodular (atomic, complete) lattice $\Pi(\mathscr{H})$ of all orthogonal projections (identified with the orthomodular lattice of all subspaces of the Hilbert space $\mathscr{H}$).

Definition 4.2. A *classical* SBZ algebra is an SBZ algebra $\mathscr{E}$ satisfying the *B*-coherence law and the following *B-compatibilty law*:

(C) For any pair of elements $a, b \in \mathscr{E}$, there exist $a_1, b_1, c \in \mathscr{E}$ such that $b_1 \perp_B c$, $a_1 \perp_B (b_1 \oplus c)$, with $a = a_1 \oplus c$ and $b = b_1 \oplus c$.

In the case of classical SBZ algebras the following result can be found in ref. 4.

Theorem 4.3. Let $\mathscr{E}$ be a classical SBZ algebra. Then, the set $\mathscr{E}_B$ of all *B*-sharp elements is a Boolean algebra.

In the fuzzy set model one has the following further result.

Theorem 4.4. Let U be a universe space and let $[0, 1]^U$ be the SBZ algebra of all fuzzy sets on U. Then the SBZ structure $\langle [0, 1]^U, \oplus, ', ^\sim, \underline{0}, \underline{1} \rangle$ is a classical SBZ algebra.

Let us recall (see Theorem 2.1) that $\langle [0, 1]^U, \leq, ', ^\sim, \underline{0}, \underline{1} \rangle$ is a distributive BZ complete lattice whose sets of *B*-sharp and *K*-sharp elements coincide with the Boolean (atomic, complete) lattice $\{0, 1\}^U$ of all characteristic functionals (identified as a Boolean lattice with the power set of the universe U).

5. FINITE-DIMENSIONAL CHARACTERIZATION OF HILBERTIAN SBZ ALGEBRAS

It is well known that, from the "sharp" point of view, the finite dimensionality of a Hilbert space $\mathscr{H}$ is characterized by the modularity of the orthocomplemented lattice of projectors.

In the BZ poset $\langle \mathscr{E}, \leq, ', ^\sim, 0, 1 \rangle$ induced from an SBZ algebra, whereas the Kleene complementation in general does not satisfy the *noncontradiction* law $[\forall a \in \mathscr{E}, a \wedge a' = 0]$ and the *exclude-middle* law $[\forall a \in \mathscr{E}, a \vee a' = 1]$, the Brouwer complementation does not satisfy the *generalized de Morgan* law:

(dM) If $a \wedge b$ exists in $\mathscr{E}$, then $a^\sim \vee b^\sim$ exists in $\mathscr{E}$ and $(a \wedge b)^\sim = a^\sim \vee b^\sim$.

Completing a partial result obtained in ref. 8, ref. 10 proves the following result:

- From the "unsharp" point of view, a Hilbert space is *finite dimensional* iff the SBZ algebra of effect operators satisfies the above (dM) law for the Brouwer complement.

Furthermore, the following result is proved in ref. 11:

- A von Neumann algebra with unit e is *finite* iff the SBZ algebra of its effects $[0, e]$ satisfies the above (dM) law for the corresponding Brouwer complement.

6. EFFECT ALGEBRAS IN PRE-HILBERT SPACES AND SBZ CHARACTERIZATION OF COMPLETENESS

Let us stress that in the context of effect algebras (without the Brouwer negation), $\langle \mathcal{E}, \oplus, ', 0, 1 \rangle$, one of the possible models is the collection $\mathcal{E}(\mathcal{H})$ of all effect operators on a *pre*-Hilbert space $\mathcal{H}$ with respect to the orthogonality relation (1), the partial sum operation (2), and the K-complementation (3), defined formally as the corresponding points (1)–(3) of the Hilbert space model discussed in Example 1.2 of Section 1.1. Thus, both the unsharp QM of effect operators on a pre-Hilbert space and (*a fortiori*) the unsharp QM of effect operators on a Hilbert space are models of the abstract effect algebra structure. It will be very important to have at this abstract level some algebraic condition which, applied to the pre-Hilbertian model, distinguishes Hilbertian situations from the other ones.

Gudder [21] introduced the notion of sharply dominating effect algebra (resp., de Morgan posets) in the following way.

Definition 6.1. An effect algebra $\mathcal{E}$ (resp., de Morgan poset $\mathcal{P}$) is said to be *sharply dominating* iff it satisfies one (and then the other) of the following two equivalent conditions:

(SD_1) $\forall a \in \mathcal{E}, \exists a^* \in \mathcal{E}_K$ (resp., $\exists a^* \in \mathcal{P}_K$): $a \leq a^*$ and if $\beta \in \mathcal{E}_K$ satisfies $a \leq \beta$, then $a^* \leq \beta$.

(SD_2) $\forall a \in \mathcal{E}, \exists a^a \in \mathcal{E}_K$ (resp., $\exists a^o \in \mathcal{P}_K$): $a^o \leq a$ and if $\gamma \in \mathcal{E}_K$ satisfies $\gamma \leq a$, then $\gamma \leq a^o$.

Conditions (SD_1) and (SD_2) define a *rough approximation space* [5]: $\langle \mathcal{E}, \mathcal{E}_K, *, {}^o \rangle$ based on $\mathcal{E}$ as the set of approximable elements, but, differently from Section 3 [compare with (BSD_1) and (BSD_2)], whose set of sharp (or crisp) elements is the collection of all K-sharp elements $\mathcal{E}_K$. These two approximation maps are in duality by the Kleene complementation $'$, i.e., $a^o = a'^{*\prime}$ and $a^o = a'^{o\prime}$. The main results of ref. 21 are the following:

Theorem 6.1. 1. If $\langle \mathcal{P}, \leq, ', 0, 1 \rangle$ is a sharply dominating de Morgan poset, then there is a *unique* B-complementation $^\sim$ on $\mathcal{P}$ [$\forall a \in \mathcal{P}$: $a^\sim := a^{*\prime}$] such that $\langle \mathcal{P}, \leq, ', {}^\sim, 0, 1 \rangle$ is a BZ poset and $\mathcal{P}_B = \mathcal{P}_K$. Conversely, if $\langle \mathcal{P}, \leq, ', {}^\sim, 0, 1 \rangle$ is a BZ-poset in which $\mathcal{P}_B = \mathcal{P}_K$, then $\mathcal{P}$ is sharply dominating and $a^* = a^{\sim\prime}$ for every $a \in \mathcal{P}$.

2. If $\langle \mathcal{E}, \oplus, ', 0, 1 \rangle$ is a sharply dominating effect algebra, then there exists a *unique* B-complementation $^\sim$ on $\mathcal{E}$ [$\forall a \in \mathcal{E}$: $a^\sim := a^{*\prime}$] such that

$\langle \mathscr{E}, \oplus, ', ^{\sim}, 0, 1 \rangle$ is an SBZ algebra and $\mathscr{E}_B = \mathscr{E}_K$. Conversely, if $\langle \mathscr{E}, \oplus, ', ^{\sim}, 0, 1 \rangle$ is an SBZ algebra in which $\mathscr{E}_B = \mathscr{E}_K$, then $\mathscr{E}$ is sharply dominating and $a^* = a^{\sim\prime}$ for every $a \in \mathscr{E}$.

As to the the collection of all fuzzy sets on the universe U, we have the following result.

Theorem 6.2. The SBZ algebra $\langle [0, 1]^U, \oplus, ', ^{\sim}, \underline{0}, \underline{1} \rangle$ of all fuzzy sets is such that

$$([0, 1]^U)_B = ([0, 1]^U)_K = \{0, 1\}^U$$

and thus is sharply dominating and for every fuzzy set $f \in [0, 1]^U$, $f^* = \chi_{A_p(f)}$, where $A_p(f) := \{x \in U: f(x) \neq 0\}$ is the *possibility domain* of f.

Now, besides the wide number of well-known possible characterizations of completeness in the class of all pre-Hilbert space structures [12, 6; and Chapter 4 of 16], we can add the following one:

Theorem 6.3. A pre-Hilbert space is complete (i.e., a Hilbert space) iff the effect algebra of all effect operators is sharply dominating.

In this case, according to the above results, there exists a *unique* B-complementation such that the structure turns out to be an SBZ effect algebra in which B-sharp elements coincide with K-sharp elements; the collection of these sharp elements is just the orthomodular lattice of all projectors:

$$\mathscr{E}_B(\mathscr{H}) = \mathscr{E}_K(\mathscr{H}) = \Pi(\mathscr{H})$$

For every effect operator F the B-complement is the projector $F^{\sim} = \mathbb{P}_{Ker(F)}$.

Let us stress that if, making use of some mathematical properties of effect operators, one needs to discriminate Hilbert spaces inside the class of all pre-Hilbert spaces, then one possibility is to take into account the Brouwer negation and the corresponding more articulate structure of SBZ algebra.

7. SBZ STRUCTURES IN AXIOMATIC UNSHARP QUANTUM MECHANICS

In the previous sections we have shown that effect operators on a Hilbert space can be naturally equipped with the structure of an SBZ algebra, producing in this way a concrete model of this algebra. Effect operators are already considered as foundamental objects of the so-called *unsharp quantum mechanics* based on Hilbert spaces. In this section we discuss an abstract *axiomatic* approach to unsharp quantum physics as studied in ref. 9, of which the standard Hilbert space realization is a concrete model.

Definition 7.1. A *preparation-effect-probability* (PEFP) structure is a triple $(\mathcal{S}, \mathcal{E}, p)$ consisting of a nonempty set $\mathcal{S}$ of *preparations*, a nonempty set $\mathcal{E}$ of *effects*, and a *probability function* $p: \mathcal{S} \times \mathcal{E} \mapsto [0, 1]$ satisfying the following conditions:

Axiom 1. There exist an effect 1, called the *certain* effect, such that

$$\forall w \in \mathcal{S}, \qquad p(w, 1) = 1$$

Axiom 2-$\mathcal{E}$. (*Indistinguishability principle of effects*). Letting $f_1, f_2 \in \mathcal{E}$, we have

$$\forall w \in \mathcal{S}, \qquad p(w, f_1) = p(w, f_2) \qquad \text{implies} \quad f_1 = f_2$$

Axiom 3. $\forall f \in \mathcal{E}, \exists f' \in \mathcal{E}$ (the *inverse* of f) such that

$$\forall w \in \mathcal{S}, \qquad p(w, f) + p(w, f') = 1$$

Axiom 4. For every *orthopair* of effects $f, g \in \mathcal{E}$ [i.e., such that $\forall w \in \mathcal{S}$, $0 \leq p(w, f) + p(w, g) \leq 1$], an effect, denoted by $f \oplus g$ and called the *sum* of f and g, exists such that for every $w \in \mathcal{S}$

$$p(w, f \oplus g) = p(w,f) + p(w, g)$$

The following is the interpretation of the primitive objects involved in an SEFP structure:

(a1) Elements from $\mathcal{S}$ are interpreted as procedures, realized by macroscopic apparatus which *prepare* both individual samples and ensembles of identical noninteracting physical objects under well-defined and repeatable conditions.

(a2) Elements from $\mathcal{E}$ are interpreted as *effects*, tested by dichotomic measuring macroscopic devices which, when interacting with a single sample of the physical entity, produce a certain definite macroscopic yes–no alternative. The occurrence of the alternative is taken as the answer "yes" and its absence as the answer "no."

(a3) For any pair $(w, f) \in \mathcal{S} \times \mathcal{E}$, the value $p(w, f) \in [0, 1]$ represents the *probability of occurrence* of the "yes" alternative for the effect f when the physical entity is prepared in w.

Note that Axiom 4 is equivalent to the following:

Axiom 4d. For every *ordered* pair of effects $f, g \in \mathcal{E}$ [i.e., such that $\forall w \in \mathcal{S}$, $p(w, f) \leq p(w, g)$], an effect, denoted by $g \ominus f$ and called the *difference* of g and f, exists such that for every $w \in \mathcal{S}$

$$p(w, f) + p(w, g \ominus f) = p(w, g)$$

In particular, if Axiom 4 is true, then the difference operation of Axiom 4d is given by $g \ominus f = (f \oplus g')'$. If Axiom 4d is true, then the sum operation of Axiom 4 is defined by $f \oplus g = (g' \ominus f)'$.

The proof of the following result can be found in ref. 9:

Theorem 7.1. Let $(\mathcal{S}, \mathcal{E}, p)$ be a PEFP structure (satisfying Axioms 1–4). Let us introduce $\mathcal{D}_{\oplus} := \{(f, g) \in \mathcal{E} \times \mathcal{E}: \forall w \in \mathcal{S}, p(w, f) + p(w, g) \leq 1\}$ and the *partial sum operation* associating with every pair $f, g \in \mathcal{D}_{\oplus}$ the (unique) effect $f \oplus g$ assured by Axiom 4. Let us consider the map $': \mathcal{E} \mapsto \mathcal{E}$, which is well defined owing to Axioms 3 and 2d.

Then the structure $\langle \mathcal{E}, \oplus, ', 0, 1\rangle$ (where $0 := 1'$) is a regular effect algebra such that

$$\exists h \in \mathcal{E}: \quad f \oplus h = g \quad \text{iff} \quad \forall w \in \mathcal{S}: \quad p(w, f) \leq p(w, h)$$

Example 7.1. The pre-Hilbert space model of PEFP structure. Let $\mathcal{H}$ be a pre-Hilbert space. Denote by $\mathcal{E}(\mathcal{H})$ the collection of all effect operators (linear operators $F: \mathcal{H} \mapsto \mathcal{H}$ satisfying the condition $\forall \psi \in \mathcal{H}$, $0 \leq \langle \psi | F\psi \rangle \leq \|\psi\|^2$) and by $\Pi(\mathcal{E}) \subseteq \mathcal{E}$ the set of projection operators. Denote by $\mathcal{S}(\mathcal{H})$ the collection of all nonzero vectors from $\mathcal{H}$. The triple $(\mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}), p)$, where $p: \mathcal{S}(\mathcal{H}) \times \mathcal{E}(\mathcal{H}) \mapsto [0, 1]$ is defined, $\forall \psi \in \mathcal{S}(\mathcal{H})$ and $\forall F \in \mathcal{E}(\mathcal{H})$, as

$$p(\psi, F) := \frac{\langle \psi | F\psi \rangle}{\|\psi\|^2} \tag{7.1}$$

is a model of a PEFP structure, i.e., a concrete mathematical structure in which the set of preparations is realized by $\mathcal{S}(\mathcal{H})$, the set of effects by $\mathcal{E}(\mathcal{H})$, and the probability function by the above mapping (7.1), and such that all Axioms 1, 2-$\mathcal{E}$, 3, and 4 are satisfied.

As usual, also in this case we have to face the problem of characterizing those pre-Hilbert PEFP structures that are based on a complete space. From the abstract point of view, we can cite the following results.

Let $(\mathcal{S}, \mathcal{E}, p)$ be an abstract PEFP structure. Then, for every effect $f \in \mathcal{E}$ we can define the *certainly-yes* and *certainly-no* domains as follows:

$$S_1(f) := \{w \in \mathcal{S}: p(x, f) = 1\} \text{ and } S_0(f) := \{w \in \mathcal{S}: p(x, f) = 0\}$$

Definition 7.2. A PEFP structure $(\mathcal{S}, \mathcal{E}, p)$ is called of *type* $\mathcal{S}$ iff the following axiom holds:

Axiom S. For any effect $f \in \mathcal{E}$, another effect $f^\nu \in \mathcal{E}$ exists such that:
(a) $S_1(f^\nu) = S_1(f)$.
(b) If $g \in \mathcal{E}$ satisfies $S_1(f^\nu) \subseteq S_1(g)$, then $f^\nu \leq g$.
(c) If $h \in \mathcal{E}$ satisfies $S_0(f^\nu) \subseteq S_0(h)$, then $h \leq f^\nu$.

The main result about type S PEFP structures is the following one [9]:

Theorem 7.2. Let $(\mathcal{S}, \mathcal{E}, p)$ be a PEFP of type $\mathcal{S}$. Then, the effect algebra $\langle \mathcal{E}, \oplus, ', 0, 1\rangle$ considered in Theorem 7.1 can be equipped with the unary operation $^{\sim}: \mathcal{E} \mapsto \mathcal{E}, f \to f^{\sim} := f'^{\nu}$, and the structure $\langle \mathcal{E}, \oplus, ', \sim, 0, 1\rangle$ is an SBZ effect algebra.

Moreover, for every effect $a \in \mathcal{E}$ the following are equivalent:

1. It is B-sharp ($a = a^{\sim\sim}$).
2. It is *open* ($a = a^{\nu}$).
3. It is S-sharp, i.e., satisfies the following conditions:

(SS_1) It is K-sharp ($a \wedge a' = 0$).
(SS_2) If $h \in \mathcal{E}$ satisfies $S_0(a) \subseteq S_0(h)$, then $h \leq a$.

For every effect $f \in \mathcal{E}$ the corresponding necessity of the inner approximation map (r3) of Section 3 is just the effect f^{ν} whose existence is assured by the above Axiom S.

According to the general theory of SBZ algebras, we can only assert that the collection $\mathcal{E}_B$ of all B-sharp effects is an FR orthoalgebra (recall Proposition 2.2). We now explore a class of PEFP structures closely related to quantum SBZ effect algebras, in the sense that the induced FR orthoalgebra is orthomodular.

Definition 7.3. A PEFP structure $(\mathcal{S}, \mathcal{E}, p)$ is said to be *finitely complete* (FC-PEFP) iff the following axioms hold:

Axiom FC. For every effect $f \in \mathcal{E}$, an effect $f^{\nu} \in \mathcal{E}$ exists such that:
(a) $S_1(f^{\nu}) = S_1(f)$.
(b) If $g \in \mathcal{E}$ satisfies $S_1(g) = S_1(f^{\nu})$, then $f^{\nu} \leq g$.
(c) If $h \in \mathcal{E}$ satisfies $S_0(h) = S_0(f^{\nu})$, then $h \leq f^{\nu}$.

Axiom 5. For every real number $\lambda \in [0, 1]$ and every effect $f \in \mathcal{E}$, there exists an effect, denoted by λf, such that for every $w \in \mathcal{S}$, $p(w, \lambda f) = \lambda p(w, f)$.

Putting together Axioms 4 and 5, one has that in any finitely complete PEFP the set of all effects is closed with respect to the convex combination: For every finite family of effects $\{f_1, \ldots, f_n\} \subseteq \mathcal{E}$ and every corresponding finite family of nonnegative real numbers $\{\lambda_{\perp}, \ldots, \lambda_h\} \subseteq \mathbb{R}_+$ such that $\Sigma_{j=1}^{n} \lambda_j = 1$, an effect $\Sigma_{j=1}^{n} \lambda_j f_j \in \mathcal{E}$ exists (the *convex combination* of the f_j with *weights* λ_j) such that

$$\forall w \in \mathcal{S}, \qquad p\left(w, \sum_{j=1}^{n} \lambda_j f_j\right) = \sum_{j=1}^{n} \lambda_j p(w, f_j) \tag{7.2}$$

In the sequel, we denote by πf_j the *product effect*, i.e., the equiweighted convex combination ($\forall_j$, $\lambda_j = 1/n$). The product effect of two effects f and g will be denoted also by $f \cdot g$.

For finitely complete PEFP structures it is possible to prove the following results [9]:

Proposition 7.1. Every FC–PEFP structure is of type S with the same necessity.

The FR orthoalgebra of all B-sharp effects induces a partial order structure of *orthomodular lattice* in which for every $a, b \in \mathscr{E}_B$, $a \wedge b = (a \cdot b)^\nu$ and $a \vee b = (a' \cdot b')'^\nu$, where $a \vee b \le a \oplus b$.

The uniqueness of the orthogonal complement holds: For every $a \in \mathscr{E}_B$, there exists a unique $a' \in \mathscr{E}_B$ such that $a \perp a'$ and $a \vee a' = 1$.

Moreover, the set of preparations *strongly determines the order* of $\mathscr{E}_B$, in the sense that for every B-sharp effect $a \in \mathscr{E}_B$ and every effect $f \in \mathscr{E}$:

$$S_1(a) \subseteq S_1(f) \Rightarrow a \le f \tag{7.3}$$

or, equivalently,

$$S_0(a) \subseteq S_0(g) \Rightarrow g \le a \tag{7.4}$$

The proof of this proposition as presented in ref. 9 (in particular for proving the orthomodularity of the lattice of B-sharp effects) strongly uses formal properties of the set $\mathscr{S}$ of preparations. It is an *open problem* whether the SBZ algebra induced according to Theorem 7.1 from a FC-PEFP satisfies the condition of B-coherence, i.e., is a quantum SBZ algebra.

Trivially, as an immediate consequence of (7.2), one has that for every convex combination (and so for every product) of effects the following holds:

$$S_1\left(\sum_{j=1}^n \lambda_j f_j\right) = \bigcap_{j=1}^n S_1(f_j) \qquad \text{and} \qquad S_0\left(\sum_{j=1}^n \lambda_j f_j\right) = \bigcap_{j=1}^n S_0(f_j) \tag{7.5}$$

As a natural extension of this property (which in particular holds for every *finite* product of effects) we can introduce the following further definition.

Definition 7.4. A finitely complete PEFP structure is called *complete* (C-PEFP) iff the following further axiom holds:

Axiom JPc. For every set $\mathscr{G}$ of effects, an effect $\pi_{\mathscr{G}} f$ exists (the *product* of $\mathscr{G}$) such that

$$S_1(\pi_{\mathscr{G}} f) = \bigcap_{f \in \mathscr{G}} S_1(f) \qquad \text{and} \qquad S_0(\pi_{\mathscr{G}} f) = \bigcap_{f \in \mathscr{G}} S_0(f)$$

With respect to this definition, the following result is given in ref. 9.

Proposition 7.2. The orthoposet structure $\langle \mathscr{E}_B, \leq, ', 0, 1\rangle$ of all B-sharp effects of a C-PEFP is an orthomodular *complete* lattice in which for every family $\mathscr{G}_B \subseteq \mathscr{E}_B$ of B-sharp effects,

$$\bigwedge_{a \in \mathscr{G}_B} a = (\pi_{\mathscr{G}_B} a)^{\nu} \qquad \text{and} \qquad \bigvee_{a \in \mathscr{G}_B} a = (\pi_{\mathscr{G}_B} a')'^{\nu}$$

Now it is possible to show that the PEFP structures of Example 7.1 are models of C-PEFP if based on Hilbert spaces, rather than (incomplete) pre-Hilbert ones.

Theorem 7.3. In the PEFP structure $\langle \mathscr{S}(\mathscr{H}), \mathscr{F}(\mathscr{H}), p\rangle$ based on a Hilbert space $\mathscr{H}$ [where p is the probability function defined by (7.1)] the following hold:

(H1) Let $F \in \mathscr{F}(\mathscr{H})$ be an effect operator. Then the orthognal projection $\mathbb{P}_{\ker(\mathbb{I}\text{-}F)}$ satisfies conditions (a), (b), and (c) of Axiom FC, Definition 7.3.

(H2) For every $\lambda \in [0, 1]$ and every effect operator $F \in \mathscr{F}(\mathscr{H})$ the operator $\lambda \cdot F$ is an effect such that $\forall\psi \in \mathscr{S}(\mathscr{H})$, $\langle\psi|(\lambda F)\psi\rangle = \lambda\langle\psi|F\psi\rangle$, i.e., Axiom 5 of Definition 7.3 is satisfied.

(H3) For every family $\mathscr{G}$ of effect operators, with the defined subspaces

$$M_1(\mathscr{G}) := \bigcap_{F \in \mathscr{G}} M_1(F) \qquad \text{and} \qquad M_0(\mathscr{G}) := \bigcap_{F \in \mathscr{G}} M_0(F)$$

the effect operator

$$\Pi_{\mathscr{G}} F := \tfrac{1}{2}(E_{M_1(\mathscr{G})} + (E_{M_0(\mathscr{G})'})$$

is such that

$$M_1(\Pi_{\mathscr{G}} F) = \bigcap_{F \in \mathscr{G}} M_1(F) \qquad \text{and} \qquad M_0(\Pi_{\mathscr{G}} F) = \bigcap_{F \in \mathscr{G}} M_0(F)$$

i.e., Axiom JPc is also satisfied.

Therefore, the Hilbert space PEFP structures are complete. It is a second *open problem* of this investigation whether the Axioms FC, 5, and JPc characterize Hilbert space PEFP with respect to the pre-Hilbert case.

7.1 The Unsharp SBZ Version of the Photon Localization Problem

Reformulating in the quantum-logic environment a no-go theorem about localization of zero mass and nonzero spin particles proved by Newton and Wigner [25], Wightman [27] showed that if the notion of *localizability* on a Hilbert space $\mathscr{H}$ involves a *PV-measure* describing whether the system is contained within a certain region in the space and *unitary representation* of the Euclidean roto-translations expressing the homogeneity and isotropy of the space by a standard *covariance* condition, then the *imprimitivity* theorem tell us "that particles of restmass zero with spin $\neq 0$ belonging to an irreducible representation of the Lorentz group do not admit position operator" [22].

An attempted solution of this problem has been proposed in refs. 22 and 1 using *POV-measures* $F: \mathcal{B}(\mathbb{R}) \mapsto \mathcal{E}(\mathcal{H})$ to describe photon localization (see also ref. 23). Since in the Jauch–Piron approach properties must be described by projectors, they define a notion of *generalized localizability* which is an *inner* PV-measure obtained, as proved in ref. 13 (see also ref. 24 for a partial result), just making the condition $(F(\Delta))^{\nu} = (F(\Delta))'^{\sim} = P_{1-\kappa\text{er}(F(\Delta))}$ of any unsharp localization effect. Let us note that for any state described by a density operator W, the mapping associated with any real Borel set Δ, the quantity $\text{Tr}[W F(\Delta)^{\nu}] \in [0, 1]$ is an inner (classical) probability measure. So, in order to formalize JP generalized localizability in the Hilbert space context, one needs a Brouwer negation.

This result can be generalized to any SBZ effect algebra once the standard notion of effect algebra *observable* is introduced; indeed, it is possible to prove that the inner approximation of any such observable gives rise to an *inner sharp observable*.

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